

# Classification of nonorientable regular embeddings of Hamming graphs

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## Abstract

By a regular embedding of a graph  $K$  in a surface we mean a 2-cell embedding of  $K$  in a compact connected surface such that the automorphism group acts regularly on flags. In this paper, we classify the nonorientable regular embeddings of the Hamming graph  $H(d, n)$ . We show that there exists such an embedding if and only if  $n = 2$  and  $d = 2$ , or  $n = 3$  or  $4$  and  $d \geq 1$ , or  $n = 6$  and  $d = 1$  or  $2$ . We also give constructions and descriptions of these embeddings.

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## 1 Introduction

A *map*  $\mathcal{M}$  is a 2-cell embedding of a graph  $K$  in a compact, connected surface  $S$ . An *automorphism* of  $\mathcal{M}$  is a permutation of its *flags* (mutually incident vertex-edge-face triples) which preserves their relations of having a vertex, edge or face in common; it therefore induces an automorphism of  $K$  which extends to a self-homeomorphism of  $S$ . The group  $G := \text{Aut}(\mathcal{M})$  of all automorphisms of  $\mathcal{M}$  acts semi-regularly on its flags, so  $|G| \leq 4|E|$ , where  $E$  is the set of edges. If this bound is attained then  $G$  acts regularly on the flags, and  $\mathcal{M}$  is called a *regular map*. Equivalently,  $\mathcal{M}$  is regular if and only if there are three involutions  $\lambda$ ,  $\rho$  and  $\tau$  in  $G$ , each fixing a distinct pair of elements  $v, e, f$  of some flag  $(v, e, f)$ ; in this case we have  $G = \langle \lambda, \rho, \tau \rangle$ . In what follows we shall assume that  $\mathcal{M}$  is a regular map, with  $\lambda$  fixing  $e$  and  $f$ , and  $\rho$  fixing  $v$  and  $f$ , so  $\tau$  fixes  $v$  and  $e$ . We call such a triple  $(\lambda, \rho, \tau)$  an *admissible triple*.

Since  $\tau\lambda = \lambda\tau$  the stabilizer  $G_e = \langle \lambda, \tau \rangle$  in  $G$  of  $e$  is a dihedral group of order 4, i.e. a Klein four-group, isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Similarly, the stabilizers  $G_v = \langle \rho, \tau \rangle$  and  $G_f = \langle \lambda, \rho \rangle$  of  $v$  and  $f$  are dihedral groups of orders  $2m$  and  $2n$ , where  $m$  is the common valency of the vertices of  $\mathcal{M}$ , i.e. the order of  $\rho\tau$ , and  $n$  is the covalency (the number of edges of each face), equal to the order of  $\lambda\rho$ .

When a regular map  $\mathcal{M}$  is represented in this way by a triple of involutions  $\lambda, \rho, \tau$  we write  $\mathcal{M} = \mathcal{M}(\lambda, \rho, \tau)$ . Two regular maps  $\mathcal{M}(\lambda, \rho, \tau)$  and  $\mathcal{M}'(\lambda', \rho', \tau')$  with underlying graphs  $K$  and  $K'$  are *isomorphic* if there is a graph isomorphism  $\psi : K \rightarrow K'$  such that  $\psi^{-1}\lambda\psi = \lambda'$ ,  $\psi^{-1}\rho\psi = \rho'$  and  $\psi^{-1}\tau\psi = \tau'$ . A detailed explanation of the above representation of regular maps can be found in [3, Theorem 3]. The basic theory of regular maps, as well as other relevant information, can also be found in [7, 8, 11, 12, 13].

If a regular map  $\mathcal{M}$  is obtained from an embedding  $i : K \rightarrow S$  of a graph  $K$  in a surface  $S$  we say that  $i$  is a *regular embedding* of  $K$ . The surface  $S$  underlying  $\mathcal{M}$  is nonorientable if and only if there is a cycle  $C$  in  $K$  with a neighbourhood in  $S$  homeomorphic to a Möbius band. Such a cycle will be called *orientation-reversing*. A regular map  $\mathcal{M}$  is nonorientable if and only if its automorphism group  $G$  is generated by  $R := \rho\tau$  and the involution  $L := \lambda\tau = \tau\lambda$ . In particular, if there is an orientation-reversing cycle  $C$  of length  $l$  in  $\mathcal{M}$  then there is an associated relation of the form  $LR^{m_1}LR^{m_2}\dots LR^{m_l} = \tau$  in  $G$ . Conversely, the existence of such a relation in  $G$  implies that  $\mathcal{M}$  is nonorientable. We call such a triple  $(\lambda, \rho, \tau)$  an *nonorientable admissible triple*.

There are only a few families of graphs for which a complete classification of their nonorientable regular embeddings is known. Such embeddings of complete graphs  $K_n$  have been classified by James [4] and Wilson [15]: these exist if and only if  $n$  is 3, 4 or 6. Nedela and the second author [10] have shown the nonexistence of a nonorientable regular embedding of the  $n$ -dimensional cube graph  $Q_n$  for all  $n$  except  $n = 2$ . In contrast with all other known cases, the complete bipartite graph  $K_{n,n}$  has a nonorientable regular embedding for infinitely many values of  $n$ , as shown by Kwak and the second author [9]: in fact, such an embedding exists if and only if  $n \equiv 2 \pmod{4}$  and all odd prime divisors of  $n$  are congruent to  $\pm 1 \pmod{8}$ .

A map  $\mathcal{M}$  is *orientably regular* if the underlying surface is orientable and the orientation-preserving subgroup  $\text{Aut}^+(\mathcal{M})$  of  $\text{Aut}(\mathcal{M})$  acts regularly on the arcs (directed edges) of  $\mathcal{M}$ . The first author [6] has recently obtained a classification of such embeddings of Hamming graphs  $H(d, n)$ , and this includes a classification of their orientable regular embeddings (called *reflexible* embeddings there). In this paper, we aim to classify the nonorientable regular embeddings of Hamming graphs. Our main result is the following theorem:

**Theorem 1.1** *There exists a nonorientable regular embedding of the Hamming graph  $H(d, n)$  if and only if either*

- (1)  $n = 2$  and  $d = 2$ , or
- (2)  $n = 3$  or  $4$  and  $d \geq 1$ , or
- (3)  $n = 6$  and  $d = 1$  or  $2$ .

*In cases (1) and (2) the embedding of  $H(d, n)$  is unique up to isomorphism, whereas in case (3) there are two such embeddings for each of the two graphs  $H(d, n)$ .*

Further information about each of these maps, namely its type, genus and automorphism group, is given in Section 2.

This paper is organized as follows. In Section 2 we construct and describe some examples of nonorientable regular embeddings of  $H(d, n)$ , and in Section 3 we classify all such embeddings by showing that each of them is isomorphic to one of these examples.

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## 2 Construction of nonorientable Hamming maps

The *Hamming graph*  $H(d, n)$  is the cartesian product of  $d$  cliques of size  $n$ . Specifically, we define  $H(d, n)$  to have vertex set  $V = [n]^d$  where  $[n] = \{0, 1, \dots, n-1\}$  for some  $n \geq 2$ , with two vertices  $u = (u_i)$  and  $v = (v_i)$  adjacent if and only if  $u_i = v_i$  for all except exactly one value of  $i$ .

The automorphism group  $\text{Aut}(H(d, n))$  of this graph is the wreath product  $S_n \wr S_d$  of the symmetric groups  $S_n$  and  $S_d$ . This is a semidirect product of a normal subgroup  $S_n \times S_n \times \dots \times S_n$ , whose  $i$ -th direct factor acts on  $i$ -th coordinate of each vertex and fixes the  $j$ -th coordinates for  $j \neq i$ , by a complement  $S_d$  which permutes the  $d$  coordinates of each vertex.

Following Coxeter and Moser [2, Ch. 8], we say that a regular map has *type*  $\{p, q\}_r$  if  $p$  is its covalency (the number of sides of each face),  $q$  is the valency of each vertex, and  $r$  is its Petrie length (the length of each Petrie polygon).

In the case  $d = 1$  the regular embeddings of  $H(d, n)$  are already known:  $H(1, n)$  is isomorphic to the complete graph  $K_n$ , and the regular embeddings of this graph have been classified by James and the first author [5] in the orientable case, and by James [4] and Wilson [15] in the nonorientable case. The results are as follows:

**Proposition 2.1** (a) *There are, up to isomorphism, just three orientable regular embeddings of complete graphs  $K_n$  for  $n \geq 2$ ; they are embeddings of a closed interval, a triangle and a tetrahedron in the sphere, with  $n = 2, 3$  and 4.*

(b) *There are, up to isomorphism, just four nonorientable regular embeddings of complete graphs  $K_n$ ; they are the antipodal quotients of a hexagon and a cube on the sphere, giving regular embeddings of  $K_3$  and  $K_4$  in the real projective plane, and the antipodal quotients of the icosahedron (on the sphere) and the great dodecahedron (on an orientable surface of genus 4), giving regular embeddings of  $K_6$  in the real projective plane and in a nonorientable surface of genus 5.*

These nonorientable regular embeddings of  $K_6$  form a Petrie dual pair, of types  $\{3, 5\}_5$  and  $\{5, 5\}_3$ ; they have automorphism group  $L_2(5) \cong A_5$ , and the second of them appears as entry N5.3 in Conder's computer-generated list of regular maps [1].

We now prove the existence part of Theorem 1.1. Case (1) is easily dealt with, as follows.

**Lemma 2.2** *There is a nonorientable regular embeddings of  $H(2, 2)$  in the real projective plane. It has type  $\{8, 2\}_8$ , and its automorphism group is a dihedral group of order 16.*

Proof: Since  $H(2, 2)$  is a cycle of length 4, we obtain a nonorientable regular embedding of this graph in the real projective plane by taking the antipodal quotient of the regular embedding of a cycle of length 8 in the sphere. The resulting map has one octagonal face, and its vertices have valency 2, so it has type  $\{8, 2\}_8$ . Its automorphism group is the symmetry group of an octagon, namely a dihedral group of order 16.  $\square$

For case (2) we use the following result, which forms part of the first author's classification [6] of the orientably regular embeddings of  $H(d, n)$ .

**Proposition 2.3** *If  $d \geq 2$  and  $n = 3$  or 4 there is an orientable regular embedding of  $H(d, n)$ . If  $n = 3$  it has type  $\{m, 2d\}_6$ , where the covalency  $m$  is  $2d$  or  $3d$  as  $d$  is even or odd. If  $n = 4$  it has type  $\{3d, 3d\}_4$ . Its automorphism group is a semidirect product of an elementary abelian normal subgroup of order  $n^d$ , acting regularly on the vertices, by a dihedral group of order  $2d(n - 1)$  fixing a vertex and acting naturally on its neighbours.*

(In fact, it is shown in [6] that these are the only orientable regular embeddings of Hamming graphs for  $n \geq 3$ .)

**Corollary 2.4** *If  $d \geq 1$  and  $n = 3$  or  $4$ , there is a nonorientable regular embedding of  $H(d, n)$ . If  $n = 3$  it has genus  $(2d - 3)3^{d-1} + 2$ , and it has type  $\{6, 2d\}_m$  where  $m$  is  $2d$  or  $3d$  as  $d$  is even or odd. If  $n = 4$  it has genus  $(3d - 4)4^{d-1} + 2$  and type  $\{4, 3d\}_{3d}$ . The automorphism group and its action on the vertices are as described in Proposition 2.3.*

Proof: Let  $d \geq 1$  and  $n = 3$  or  $4$ , and let  $\mathcal{M}$  be the orientable regular embedding of  $H(d, n)$  given by Proposition 2.3. Since Hamming graphs are not bipartite for  $n \geq 3$ , the Petrie dual  $\mathcal{N} = P(\mathcal{M})$  of  $\mathcal{M}$  is a nonorientable regular embedding of  $H(d, n)$ . Petrie duality transposes covalency and Petrie length, so the type of  $\mathcal{N}$  is as claimed, and the genus then follows from the Euler formula. Since Petrie duality preserves the automorphism group and its action on vertices,  $\text{Aut } \mathcal{N}$  is as in Proposition 2.3.  $\square$

For example, consider the first nontrivial case, namely  $d = 2$ . If  $n = 3$  then  $\mathcal{N}$  has genus 5 and type  $\{6, 4\}_4$ , and is the dual of entry N5.2 in Conder's list [1] of nonorientable regular maps; if  $n = 4$  then  $\mathcal{N}$  has genus 10 and type  $\{4, 6\}_6$ , and is entry N10.1 in this list. If  $d = 3$  then for  $n = 3$  we obtain N29.2, of genus 29 and type  $\{6, 6\}_9$ , and for  $n = 4$  we have N82.1, of type  $\{4, 9\}_9$  and genus 82.

The rest of this section is devoted to case (3) of Theorem 1.1, so we take  $n = 6$ .

For  $d = 1$ , the Hamming graph  $H(1, 6)$  is the complete graph  $K_6$ . It is known by work of James [4] and Wilson [15] that there are, up to isomorphism, two nonorientable regular embeddings of  $K_6$ , described in Proposition 2.1(b) and the remarks following it. Here we will give a group-theoretic construction of a Petrie dual pair of nonorientable regular embeddings of  $H(2, 6)$ .

Just as the nonorientable regular embeddings of  $H(1, 6)$  can be obtained from the action of  $A_5$  by conjugation on its six Sylow 5-subgroups, those of  $H(2, 6)$  can be obtained from the corresponding action of the group  $P := PGL_2(9)$ . This has a simple subgroup  $L := L_2(9) \cong A_6$  of index 2, which has twelve icosahedral subgroups  $A \cong A_5$ , forming two conjugacy classes  $I$  and  $J$  of size six: those in  $I$  are the point stabilisers in the natural action of  $A_6$ , while those in  $J$  act transitively as  $L_2(5)$  in its natural action on the six points of the projective line  $\mathbb{P}^1(5)$  over the field  $\mathbb{F}_5$ . These two classes are transposed by conjugation by elements of  $P \setminus L$ , so they merge to form a single conjugacy class in  $P$ . Each of the 36 Sylow 5-subgroups  $S \cong C_5$  of  $P$  lies in exactly one group  $A \in I$  and one group  $B \in J$ , so by defining two Sylow 5-subgroups to be adjacent if they are contained in a common icosahedral subgroup, we obtain a graph  $H \cong H(2, 6)$  on which  $P$  acts by conjugation as a group of automorphisms. The stabiliser of a vertex  $S$  is its normaliser  $N_P(S)$  in  $P$ , a dihedral group  $D$  of order

20 which acts transitively on the ten neighbours of  $S$ . We will use this to construct nonorientable regular embeddings of  $H(2, 6)$ .

Let  $\lambda, \rho$  and  $\tau$  be the elements of  $P$  corresponding to the matrices

$$M_\lambda = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_\rho = \begin{pmatrix} 0 & 1+i \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad M_\tau = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in  $GL_2(9)$ , where  $\mathbb{F}_9 = \mathbb{F}_3(i)$  with  $i^2 = -1$ . The elements  $\rho$  and  $\tau$  generate a dihedral group  $D$  of order 20, which is maximal in  $P$  and does not contain  $\lambda$ , so these three elements generate  $P$ . They satisfy

$$\lambda^2 = \rho^2 = \tau^2 = (\lambda\tau)^2 = 1,$$

so they determine a regular map  $\mathcal{M}$  with  $\text{Aut}(\mathcal{M}) \cong \mathcal{P}$ : the vertices, edges and faces correspond to the cosets in  $P$  of the subgroups  $\langle \rho, \tau \rangle$ ,  $\langle \lambda, \tau \rangle$  and  $\langle \lambda, \rho \rangle$ , with incidence given by non-empty intersection. Since  $\lambda\rho$ ,  $\rho\tau$  and  $\lambda\rho\tau$  have orders 10, 10 and 8 respectively,  $\mathcal{M}$  has type  $\{10, 10\}_8$ , while its Petrie dual  $\mathcal{N}$  has type  $\{8, 10\}_{10}$ . Since  $|P| = 720$ ,  $\mathcal{M}$  and  $\mathcal{N}$  have 36 and 45 faces, respectively, so they have Euler characteristics  $\chi = -108$  and  $-99$ . Since  $\det M_\lambda = 1$ ,  $\lambda$  is contained in the unique subgroup  $L$  of index 2 in  $P$ , so  $\mathcal{M}$  and  $\mathcal{N}$  are nonorientable and hence have genera  $2 - \chi = 110$  and  $101$  (they appear as entries N110.7 and N101.8 in [1]). These two maps have the same underlying graph  $K$ , which we will show is isomorphic to  $H$ .

The subgroup  $D = \langle \rho, \tau \rangle$  stabilising a vertex of  $K$  is the normaliser in  $P$  of the Sylow 5-subgroup  $S = \langle a \rangle$ , where  $a = (\rho\tau)^2$ ; thus the vertices of  $K$  can be identified with the 36 Sylow 5-subgroups of  $P$ , permuted by conjugation, and hence with the vertices of  $H$ . Now we consider the edges of  $K$ . The Sylow 5-subgroup  $S$  corresponds to the vertex  $D$  of  $K$ , and this is adjacent in  $K$ , through their common edge  $\langle \lambda, \tau \rangle$ , to the vertex corresponding to the Sylow 5-subgroup  $T = S^\lambda$  generated by the element  $b = a^\lambda$ . The subgroup  $A$  generated by  $a$  and  $b$  is also generated by  $a$  and  $a^2b$ , which have orders 5 and 3, while their product has order 2; thus  $A$  is an epimorphic image of the triangle group of type  $(5, 3, 2)$ , isomorphic to the simple group  $A_5$ , so  $S$  and  $T$  generate an icosahedral subgroup of  $P$ . Thus  $S$  and  $T$  are neighbours in  $H$ , and since  $P$  acts edge transitively on both  $K$  and  $H$ , it follows that their edge sets correspond. Thus our identification of their vertex sets gives an isomorphism  $K \rightarrow H$  commuting with the actions of  $P$  on these two graphs. Since  $\mathcal{M}$  and  $\mathcal{N}$  were constructed as nonorientable regular embeddings of  $K$ , this shows that they also yield such embeddings of  $H \cong H(2, 6)$ .

To summarise, we have proved the following result.

**Lemma 2.5** *There is a Petrie dual pair of nonisomorphic nonorientable regular embeddings of  $H(2, 6)$ . They have genera 110 and 101, and types  $\{10, 10\}_8$  and  $\{8, 10\}_{10}$ . Their automorphism group is isomorphic to  $PGL_2(9)$ .*

In the next section, we will show that the regular maps constructed in this section are, up to isomorphism, the only nonorientable regular embeddings of Hamming graphs.

### 3 Classification of nonorientable Hamming maps

In this section, we classify the nonorientable regular embeddings of Hamming graphs  $H(d, n)$  by completing the proof of Theorem 1.1. The case  $d = 1$ , where  $H(1, n) \cong K_n$ , is covered by the work of James [4] and Wilson [15] summarised in Proposition 2.1(b), showing that  $n = 3, 4$  or  $6$ , so from now on we will assume that  $d \geq 2$ . If  $n = 2$  then  $H(d, n)$  is isomorphic to the  $d$ -cube  $Q_d$ , and it is known from work of Nedela and the second author [10] that the only nonorientable regular embedding of  $Q_d$  for any  $d \geq 2$  is the embedding of  $Q_2 \cong H(2, 2)$  described in Lemma 2.2. This deals with the case  $n = 2$ , including part (1) of Theorem 1.1, so we will assume from now on that  $n \geq 3$ .

For notational convenience we will regard  $[n] = \{0, \dots, n-1\}$  as the ring  $\mathbb{Z}_n$ , and we will label the coordinate places of vertices with elements  $i \in [d] := \{0, 1, \dots, d-1\}$ . We will identify  $\text{Aut}(H(d, n))$  with  $S_n \wr S_d$ , acting as described at the beginning of Section 2, and we will write each element of this group in the form  $\delta = \delta^* \delta'$  where  $\delta^* = (\delta_0, \dots, \delta_{d-1}) \in S_n^d$  with each  $\delta_i \in S_n$ , and  $\delta' \in S_d$ . We will use the facts that the projection  $\delta \mapsto \delta'$  is a homomorphism, and that the elements  $\delta$  with each  $\delta_i$  even form a subgroup  $A_n \wr S_d$  of  $S_n \wr S_d$ .

Let us define permutations

$$\alpha_d = (0 \ 1 \ \dots \ d-1) \quad \text{and} \quad \beta_d = (0)(1 \ d-1)(2 \ d-2) \dots$$

in  $S_d$  (so that  $\beta_d$  is the identity permutation  $id$ ), and

$$\beta_n = (0)(1 \ n-1)(2 \ n-2) \dots \quad \text{and} \quad \gamma_n = (1 \ 2 \ \dots \ n-1)$$

in  $S_n$ . Let  $O$  denote the vertex  $(0, \dots, 0)$ , and for each  $i = 0, \dots, d-1$  let  $e_i$  denote the vertex  $(0, \dots, 0, 1, 0, \dots, 0)$  having a single non-zero coordinate 1 in position  $i$ . By using the natural structure of the vertex set  $\mathbb{Z}_n^d$  as a  $\mathbb{Z}_n$ -module, we can denote a general vertex  $(v_0, v_1, \dots, v_{d-1})$  of  $H(d, n)$  by  $\sum_{i=0}^{d-1} v_i e_i$ .

**Lemma 3.1** *For any admissible triple  $(\lambda, \rho, \tau)$ , corresponding to an orientable or nonorientable regular embedding of  $H(d, n)$  with  $d \geq 2$  and  $n \geq 3$ , there exists an automorphism  $\phi$  of  $H(d, n)$  such that*

$$\begin{aligned} \phi^{-1} \tau \phi &= ((0)(1)(2 \ n-1)(3 \ n-2) \dots, \beta_n, \dots, \beta_n) \beta_d \\ \phi^{-1} \rho \tau \phi &= (id, id, \dots, \gamma_n) \alpha_d \\ \phi^{-1} \lambda \tau \phi &= (\sigma_0, \sigma_1, \dots, \sigma_{d-1}) \theta, \end{aligned}$$

where  $0^\theta = 0$ ,  $\theta^2 = id$ ,  $0^{\sigma_0} = 1$ ,  $1^{\sigma_0} = 0$ , and  $0^{\sigma_i} = 0$  and  $\sigma_i \sigma_{i^\theta} = id$  for each  $i \in [d] \setminus \{0\}$ . In particular, if  $(\lambda, \rho, \tau)$  is a nonorientable admissible triple for  $H(d, n)$ , we can assume that  $\theta = \beta_d$  in the above form. The subgroup  $\langle (\rho\tau)^d, \lambda\tau, \tau \rangle$  is isomorphic to the automorphism group of a regular embedding of  $K_n$ , which is nonorientable if  $d \geq 3$ .

Proof: Let  $(\lambda, \rho, \tau)$  be an admissible triple for  $H(d, n)$ , generating the automorphism group  $G$  of a regular embedding  $\mathcal{M}$  of  $H(d, n)$ . Thus  $\rho$  and  $\tau$  fix a vertex  $v$ , which  $\lambda$  transposes with an adjacent vertex  $w$  fixed by  $\tau$ . Since  $\text{Aut}(H(d, n)) = S_n \wr S_d$  acts transitively on the arcs of  $H(d, n)$ , there is a graph automorphism  $\phi$  sending  $v$  and  $w$  to  $O$  and  $e_0$ . Then  $\lambda_1 := \phi^{-1}\lambda\phi$  transposes  $O$  and  $e_0$ , while  $\rho_1 := \phi^{-1}\rho\phi$  fixes  $O$ , and  $\tau_1 := \phi^{-1}\tau\phi$  fixes  $O$  and  $e_0$ . The conjugate admissible triple  $(\lambda_1, \rho_1, \tau_1)$  generates the automorphism group  $G_1 := \phi^{-1}G\phi$  of a regular embedding  $\mathcal{M}_1 = \phi(\mathcal{M})$  of  $H(d, n)$ . Now  $\langle \rho\tau \rangle$  acts transitively on the  $d(n-1)$  neighbours of  $v$ ; these form  $d$  disjoint cliques, which are blocks of imprimitivity for this group, so  $(\rho\tau)'$  is a cyclic permutation of  $[d]$ , and we may choose  $\phi$  so that  $(\rho_1\tau_1)' = \alpha_d$ . More specifically, we may choose  $\phi$  (or equivalently relabel the vertices) so that  $\lambda_1, \rho_1$  and  $\tau_1$  satisfy

- (i)  $\rho_1\tau_1 = (id, id, \dots, \gamma_n)\alpha_d$ , so that for any  $k \in \mathbb{Z}_n \setminus \{0\}$  we have  $(ke_i)^{\rho_1\tau_1} = ke_{i+1}$  for all  $i = 0, \dots, d-2$ , while  $(ke_{d-1})^{\rho_1\tau_1}$  is  $(k+1)e_0$  or  $e_0$  as  $k \neq n-1$  or  $k = n-1$  respectively,
- (ii)  $\lambda_1$  transposes the vertices  $O$  and  $e_0$ , and preserves the face incident with  $O, e_0$  and  $e_{n-1}$ ,
- (iii)  $\rho_1$  fixes the vertex  $O$  and preserves the face incident with  $O, e_0$  and  $e_{n-1}$ , and
- (iv)  $\tau_1$  fixes both  $O$  and  $e_0$ .

Since  $\tau_1$  inverts  $\rho_1\tau_1$  by conjugation, and commutes with  $\lambda_1$ , one can easily verify that

$$\begin{aligned}\tau_1 &= ((0)(1)(2 \ n-1)(3 \ n-2) \cdots, \beta_n, \dots, \beta_n)\beta_d \\ \lambda_1\tau_1 &= (\sigma_0, \sigma_1, \dots, \sigma_{d-1})\theta,\end{aligned}$$

where  $0^\theta = 0$ ,  $\theta^2 = id$ ,  $0^{\sigma_0} = 1$ ,  $1^{\sigma_0} = 0$ , and  $0^{\sigma_i} = 0$  and  $\sigma_i \sigma_{i^\theta} = id$  for all  $i \in [d] \setminus \{0\}$ . Note that if  $d = 2$  then both  $\beta_d$  and  $\theta$  are the identity permutation.

Let  $R_1 := \rho_1\tau_1$  and  $L_1 := \lambda_1\tau_1$ . The subgraph  $K$  of  $H(d, n)$  induced by the set of vertices  $\{ke_0 \mid k \in \mathbb{Z}_n\}$  is isomorphic to the complete graph  $K_n$ . It is invariant under  $R_1^d$ ,  $L_1$  and  $\tau_1$ , and hence under the subgroup  $H = \langle R_1^d, L_1, \tau_1 \rangle$  of  $G_1$  which they generate. Now  $H$  acts transitively (and hence regularly) on the flags of  $\mathcal{M}_1$  incident with  $K$ ,



with  $(\lambda_1, R_1^d \tau_1, \tau_1)$  an admissible triple, so  $H$  is isomorphic to the automorphism group of a regular embedding  $\mathcal{K}$  of  $K_n$ . (In fact, we have constructed  $\mathcal{K}$  from  $\mathcal{M}_1$  by applying Wilson's map operation  $H_d$  [14], which raises the local rotation of arcs around each vertex to its  $d$ -th power; in this case  $d$  is not coprime to the valency  $d(n-1)$ , so  $H$  is not transitive on the flags of  $\mathcal{M}_1$ , and we have taken a single orbit to define  $\mathcal{K}$ .)

Since  $n \geq 3$  the graph  $K$  contains a cycle of length 3, so there exist  $k_1, k_2, k_3 \in \mathbb{Z}_{n-1} \setminus \{0\}$  such that the element

$$\delta := L_1 R_1^{dk_1} L_1 R_1^{dk_2} L_1 R_1^{dk_3}$$

of  $H$  is in  $\langle \tau_1 \rangle$ . Now

$$R_1^d = (\gamma_n, \dots, \gamma_n) \in S_n \times \dots \times S_n,$$

so  $\delta$  has image  $\delta' = \theta^3 = \theta$  in  $S_d$ .

We will assume that  $(\lambda, \rho, \tau)$  is a nonorientable admissible triple, so the triple  $(\lambda_1, \rho_1, \tau_1)$  is also nonorientable. Thus  $\tau_1 \in \langle R_1, L_1 \rangle$ , so by taking images in  $S_d$  we have  $\tau'_1 \in \langle R'_1, L'_1 \rangle$ , that is,  $\beta_d \in \langle \alpha_d, \theta \rangle$ . If  $d \geq 3$  then visibly  $\beta_d \notin \langle \alpha_d \rangle$ , so  $\theta \neq id$ . Thus  $\delta' \neq id$ , so  $\delta \neq id$  and hence  $\delta = \tau_1$  (and  $\theta = \tau'_1 = \beta_d$ ). Thus  $\tau_1$  is contained in the subgroup  $\langle R_1^d, L_1 \rangle$  of  $H$ , so  $\langle R_1^d, L_1 \rangle = H$  and  $\mathcal{K}$  is a nonorientable regular embedding of  $K_n$ .  $\square$

It is thus sufficient to consider nonorientable admissible triples  $(\lambda, \rho, \tau)$  for  $H(d, n)$  of the form given by  $(\lambda_1, \rho_1, \tau_1) = (\phi^{-1}\lambda\phi, \phi^{-1}\rho\phi, \phi^{-1}\tau\phi)$  in Lemma 3.1. For any such triple  $(\lambda, \rho, \tau)$ , the group  $H = \langle (\rho\tau)^d, \lambda\tau, \tau \rangle$  is isomorphic to the automorphism group of a regular embedding  $\mathcal{K}$  of  $K_n$ , which is nonorientable if  $d \geq 3$ , but may be orientable or nonorientable if  $d = 2$ . In either case, since we are assuming that  $n \geq 3$ , Proposition 2.1 implies the following:

**Corollary 3.2** *If there is a nonorientable regular embedding of  $H(d, n)$  with  $d \geq 2$  and  $n \geq 3$ , then  $n = 3, 4$  or  $6$ .*

To deal with case (2) of Theorem 1.1, let us assume that  $n = 3$  or  $4$ .

**Lemma 3.3** *If  $d \geq 2$  and  $n = 3$  or  $4$ , there is, up to isomorphism, at most one nonorientable regular embedding of  $H(d, n)$ .*

Proof: Let  $n = 3$  and  $d = 2$ . Then  $\tau = (id, (1 \ 2))id$ ,  $R = (id, (1 \ 2))(0 \ 1)$  and  $L = ((0 \ 1), \sigma_1)id$ , where  $\sigma_1 = id$  or  $(1 \ 2)$ . One can check that  $LR^2LR^2LR^2 = \tau$  or  $id$  as  $\sigma_1 = id$  or  $(1 \ 2)$  respectively. By Proposition 2.3 there is an orientable regular

embedding of  $H(2, 3)$ , which must correspond to the case  $\sigma_1 = (1 \ 2)$ , so there is at most one nonorientable regular embedding of  $H(2, 3)$ , with  $\sigma_1 = id$ .

Now let  $n = 3$  and  $d \geq 3$ . Then  $\sigma_0 = (0 \ 1)$ , and for each  $i = 1, 2, \dots, d-1$  the element  $\sigma_i = \sigma_{d-i}$  is the identity or  $(1 \ 2)$  because  $0^{\sigma_i} = 0$ . By Lemma 3.1, the subgroup  $\langle (\rho\tau)^d, \lambda\tau \rangle$  is isomorphic to the automorphism group of a nonorientable regular embedding  $\mathcal{K}$  of  $K_3$ . By Proposition 2.1(b) the underlying surface of  $\mathcal{K}$  is the real projective plane, and the graph  $K_3$  has a neighbourhood homeomorphic to a Möbius band. Thus  $LR^d LR^d LR^d = \tau$ , which implies that

$$\sigma_i(1 \ 2)\sigma_{d-i}(1 \ 2)\sigma_i(1 \ 2) = \beta_3 = (1 \ 2)$$

for each  $i = 1, 2, \dots, d-1$ , and hence  $\sigma_i(1 \ 2)\sigma_{d-i}(1 \ 2)\sigma_i = id$ . This implies that  $\sigma_i = \sigma_{d-i} = id$ , so there is at most one nonorientable regular embedding of  $H(d, 3)$  for each  $d \geq 3$ .

Next let  $n = 4$  and  $d = 2$ . Then  $\tau = ((2 \ 3), (1 \ 3))id$ ,  $R = (id, (1 \ 2 \ 3))(0 \ 1)$  and  $L = (\sigma_0, \sigma_1)id$ . Now  $\tau \in \langle R, L \rangle$ ; since the coordinates of the element  $\tau^* \in S_4 \times S_4$  are both odd permutations, whereas those of  $R^*$  are both even, each  $\sigma_i$  must be odd. Since  $0^{\sigma_1} = 0$ , it follows that  $\sigma_1$  is a transposition. By Lemma 3.1 the subgroup  $H = \langle R^2, L, \tau \rangle$  is isomorphic to the automorphism group of a regular embedding  $\mathcal{K}$  of  $K_4$ . Now  $K_4$  contains a 3-cycle, so there exist  $k_1, k_2, k_3 \in \mathbb{Z}_3 \setminus \{0\}$  such that the element

$$\delta := LR^{2k_1} LR^{2k_2} LR^{2k_3}$$

is in  $\langle \tau \rangle$ . Since the second coordinate of  $\delta^*$  is an odd permutation,  $\delta = \tau$ , so  $\tau \in \langle R^2, L \rangle$  and  $\mathcal{K}$  is nonorientable. Thus  $\mathcal{K}$  is the unique nonorientable regular embedding of  $K_4$ , namely the antipodal quotient of the cube. This implies that  $\sigma_0 = (0 \ 1)$  and  $LR^2 LR^4 LR^2 = \tau$ . Comparing the second coordinates of  $(LR^2 LR^4 LR^2)^*$  and  $\tau^*$ , we have  $\sigma_1(1 \ 2 \ 3)\sigma_1(1 \ 3 \ 2)\sigma_1(1 \ 2 \ 3) = (1 \ 3)$ . Among the three possible transpositions  $\sigma_1 \in S_4$  fixing 0, one easily checks that only  $(1 \ 3)$  satisfies this equation, so there is at most one nonorientable regular embedding of  $H(2, 4)$ .

Finally let  $n = 4$  and  $d \geq 3$ . By Lemma 3.1,  $\langle R^d, L \rangle$  is isomorphic to the automorphism group of a nonorientable regular embedding of  $K_4$ , which again implies that  $\sigma_0 = (0 \ 1)$  and  $LR^d LR^{2d} LR^d = \tau$ . Hence for each  $i = 1, 2, \dots, d-1$ ,

$$\sigma_i(1 \ 2 \ 3)\sigma_{d-i}(1 \ 3 \ 2)\sigma_i(1 \ 2 \ 3) = \beta_4 = (1 \ 3).$$

Since  $\sigma_i\sigma_{d-i} = id$ , both  $\sigma_i$  and  $\sigma_{d-i}$  are odd permutations. Since they fix 0, this implies that both  $\sigma_i$  and  $\sigma_{d-i}$  are transpositions, and hence  $\sigma_i = \sigma_{d-i}$ . Now one can easily show that  $\sigma_i = \sigma_{d-i} = (1 \ 3)$ , so there is at most one nonorientable regular embedding

of  $H(d, 4)$  for each  $d \geq 3$ . □

This result and Corollary 2.4 deal with the cases  $n = 3$  and  $n = 4$ , including part (2) of Theorem 1.1. By Corollary 3.2, in order to complete the proof, and to deal with part (3), we may assume from now on that  $n = 6$ .

**Lemma 3.4** *There is no nonorientable regular embedding of  $H(d, 6)$  for any  $d \geq 3$ , and there are, up to isomorphism, at most two nonorientable regular embeddings of  $H(2, 6)$ .*

*Proof:* We will use the easily verified fact that the two nonorientable regular embeddings of  $H(1, 6) \cong K_6$  described in Proposition 2.1(b) are derived from admissible triples  $(\lambda, \rho, \tau)$  for this graph with

$$\tau = (2 \ 5)(3 \ 4), \quad R = \gamma_6 = (1 \ 2 \ 3 \ 4 \ 5) \quad \text{and} \quad L = (0 \ 1)(2 \ 5) \quad \text{or} \quad (0 \ 1)(3 \ 4)$$

in  $S_6$ . If  $L = (0 \ 1)(2 \ 5)$  then  $LR^4LR^2LR^2 = \tau$ , which implies that there is a 4-cycle in  $K_6$  with a neighbourhood homeomorphic to a Möbius band. Since these two nonorientable embeddings form a Petrie dual pair, it follows that both embeddings have such a 4-cycle.

Now let  $(\lambda, \rho, \tau)$  be a nonorientable admissible triple for  $H(d, 6)$ , with  $d \geq 2$ . By Lemma 3.1, we can assume that

$$\begin{aligned} \tau &= ((2 \ 5)(3 \ 4) \cdots, \beta_6, \dots, \beta_6)\beta_d \\ R &= \rho\tau = (id, id, \dots, \gamma_6)\alpha_d \\ L &= \lambda\tau = (\sigma_0, \sigma_1, \dots, \sigma_{d-1})\beta_d. \end{aligned}$$

Moreover the subgroup  $H = \langle R^d, L, \tau \rangle$  is isomorphic to the automorphism group of a regular embedding  $\mathcal{K}$  of the subgraph  $K \cong K_6$  induced by the vertices  $ke_0$  for  $k \in \mathbb{Z}_6$ . Proposition 2.1 shows that  $\mathcal{K}$  must be nonorientable, so by our earlier argument there is a 4-cycle in  $K$  with a neighbourhood in  $\mathcal{K}$  homeomorphic to a Möbius band. Thus there exist  $k_1, k_2, k_3$  and  $k_4$  such that  $LR^{dk_1}LR^{dk_2}LR^{dk_3}LR^{dk_4} = \tau$ . By taking images in  $S_d$  we see that  $\beta_d = id$ , and hence  $d = 2$ . Thus there is no nonorientable regular embedding of  $H(d, 6)$  for any  $d \geq 3$ .

Finally, let  $d = 2$ . By considering the actions of  $R^2 = (\gamma_6, \gamma_6)$  and  $L$  on the coefficients  $k \in \mathbb{Z}_6$  of the vertices  $ke_0$  of  $K$  we see that  $\gamma_6$  and  $\sigma_0$  generate a subgroup of  $S_6$  isomorphic to  $\text{Aut } \mathcal{K}$ . The two regular embeddings of  $K_6$  in Proposition 2.1(b) have automorphism groups isomorphic to  $L_2(5)$ , so  $\langle \gamma_6, \sigma_0 \rangle \cong L_2(5)$ . This group is perfect,

so  $\sigma_0$  is an even permutation; since  $\sigma_0$  is an involution, it is a double transposition. Now  $\sigma_0$  transposes 0 and 1, so there are just six possibilities, and inspection shows that  $\langle \gamma_6, \sigma_0 \rangle \cong L_2(5)$  if and only if  $\sigma_0 = (0\ 1)(2\ 5)$  or  $(0\ 1)(3\ 4)$ .

If  $\sigma_0 = (0\ 1)(2\ 5)$  then  $(LR^2)^3 = id$ , so  $\mathcal{K}$  has covalency 3; by Proposition 2.1(b) it is therefore the antipodal quotient of the icosahedral map. This satisfies  $LR^4LR^6LR^4 = \tau$ , and these two equations imply that  $\sigma_1$  must satisfy

$$(\sigma_1(1\ 2\ 3\ 4\ 5))^3 = id$$

and

$$\sigma_1(1\ 3\ 5\ 2\ 4)\sigma_1(1\ 4\ 2\ 5\ 3)\sigma_1(1\ 3\ 5\ 2\ 4) = (2\ 5)(3\ 4).$$

By the first equation,  $\sigma_1$  is an even permutation. Note also that  $\sigma_1$  is an involution fixing 0. One can check that  $(1\ 2)(4\ 5)$  is the only permutation in  $A_6$  satisfying these conditions on  $\sigma_1$ , so there is at most one nonorientable regular embedding of  $H(2, 6)$  with  $\sigma_0 = (0\ 1)(2\ 5)$ .

If  $\sigma_0 = (0\ 1)(3\ 4)$  then a similar argument, in which  $\mathcal{K}$  is now the antipodal quotient of the great dodecahedron, shows that the only possibility for  $\sigma_1$  is  $(1\ 4)(2\ 5)$ , so there is at most one nonorientable regular embedding of  $H(2, 6)$  with  $\sigma_0 = (0\ 1)(3\ 4)$ .  $\square$

In the case  $d = 2$  of the above proof, if  $\sigma_0 = (0\ 1)(2\ 5)$  and  $\sigma_1 = (1\ 2)(4\ 5)$  the covalency of the corresponding embedding of  $H(2, 6)$  is 8, and if  $\sigma_0 = (0\ 1)(3\ 4)$  and  $\sigma_1 = (1\ 4)(2\ 5)$  it is 10, so these embeddings are respectively the maps  $\mathcal{N}$  and  $\mathcal{M}$  of types  $\{8, 10\}_{10}$  and  $\{10, 10\}_8$  constructed in the proof of Lemma 2.5.

This deals with part (3) of Theorem 1.1, and completes the proof of this result.

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